*Journal of Statistical Physics, Vol. 69, Nos. 3/4, 1992*

# Long-Time Tails in a Random Diffusion Model

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*Received February 25, 1992*

Let $w = \{w(x): x \in \mathbf{Z}^d\}$ be a positive random field with i.i.d. distribution $\mu$. Given its realization, let $X_t$ be the position at time $t$ of a particle starting at the origin and performing a simple random walk with jump rate $w^{-1}(X_t)$. The process $X = \{X_t: t \geqslant 0\}$ combined with $w$ on a common probability space is an example of random walk in random environment. We consider the quantities $\Delta_t = (d/dt)\,\mathbf{E}_\mu(X_t^2 - M^{-1}t)$ and $\Delta_t(w) = (d/dt)\,\mathbf{E}_w(X_t^2 - M^{-1}t)$. Here $\mathbf{E}_w$ is expectation over $X$ at fixed $w$ and $\mathbf{E}_\mu = \int \mathbf{E}_w\, \mu(dw)$ is the expectation over both $X$ and $w$. We prove the following long-time tail results: (1) $\lim_{t\to\infty} t^{d/2}\Delta_t = V^2 M^{d/2-3}(d/2\pi)^{d/2}$ and (2) $\lim_{t\to\infty} t^{d/4}\Delta_{st}(w) = Z_s$ weakly in path space, with $\{Z_s: s>0\}$ the Gaussian process with $\mathbf{E}Z_s = 0$ and $\mathbf{E}Z_r Z_s = V^2 M^{d/2-4}(d/2\pi)^{d/2}$ $(r+s)^{-d/2}$. Here $M$ and $V^2$ are the mean and variance of $w(0)$ under $\mu$. The main surprise is that fixing $w$ changes the power of the long-time tail from $d/2$ to $d/4$. Since $\Delta_t = M\mathbf{E}_{\mu_0}([w^{-1}(X_0) - M^{-1}][w^{-1}(X_t) - M^{-1}])$, with $\mu_0$ the stationary measure for the environment process, our result (1) exhibits a long-time tail in an equilibrium autocorrelation function.

**KEY WORDS:** Random walk in random environment; long-time tail; environment process; local times; spectral theorem; Tauberian theorem; functional central limit theorem.

## INTRODUCTION AND STATEMENT OF RESULTS

In *Lorentz models* a particle randomly moves in $\mathbf{R}^d$ or $\mathbf{Z}^d$ while interacting with a static random environment that influences its motion. One of the physically interesting quantities is the *equilibrium velocity autocorrelation function* of the particle, which is believed to decay like

$$\mathbf{E}(V_0 V_t) \sim -At^{-d/2-1} \qquad (t \to \infty) \tag{0.1}$$

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0022-4715/92/1100-0731$06.50/0 

with $A>0$ some model-dependent amplitude. Here $\mathbf{E}$ is the expectation over motion and environment, and $V_t$ is the velocity of the particle at time $t$. Such slow decay is called a *long-time tail* in the physics literature.[13]

If $X_t=\int_0^t V_s\,ds$ is the displacement of the particle at time $t$ and $\{V_t: t\geqslant 0\}$ is a *stationary process*, then

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2 \mathbf{E}X_t^2=\mathbf{E}(V_0 V_t) \tag{0.2}$$

This shows that the long-time tail also exhibits itself as a correction to the diffusive behavior

$$\mathbf{E}X_t^2 \sim Dt \qquad (t\to\infty) \tag{0.3}$$

with $D$ the diffusion constant.

The origin of long-time tails in equilibrium autocorrelation functions is well understood heuristically. The slow decay arises because the particle may return to sites visited earlier and recognize the medium. This induces a memory effect, which is governed by slowly decaying return probabilities that are typical for diffusive motion. However, such an explanation is rather vague, and indeed it seems to be very difficult to get mathematically precise results in concrete models. To our knowledge, refs. 2, 3, 6, 7, and 12 are the only papers where long-time-tail results are proved rigorously.

In the present paper we study the so-called *random waiting time model*.[5] Let

$$w=\{w(x): x\in\mathbf{Z}^d\} \tag{0.4}$$

be a collection of random variables with values in $(0,\infty)$ and with joint distribution $\mu$ satisfying

$$\begin{aligned}&\text{(i)}\quad \mu \text{ is stationary and ergodic under translations}\\ &\text{(ii)}\quad \int w^{-1}(0)\,\mu(dw)<\infty\\ &\text{(iii)}\quad \int w^{2}(0)\,\mu(dw)<\infty\end{aligned} \tag{0.5}$$

The collection $w$ is called the environment of waiting times. Given $w$, let

$$X=\{X_t: t\geqslant 0\} \qquad (X_0=0) \tag{0.6}$$

be the continuous-time Markov process on $\mathbf{Z}^d$ with generator

$$L_w f(x)=\frac{1}{w(x)}\frac{1}{2d}\sum_{y:\,|y-x|=1}[f(y)-f(x)] \tag{0.7}$$

i.e., simple random walk with jump rate $w^{-1}(X_t)$. Then the random waiting time model is defined as the combined process $(X, w)$. This is an example of *random walk in random environment.*

If $w$ is distributed according to $\mu$, then the particle is not "in equilibrium with its environment," i.e.,

$$W_t\colon \quad x \to w(x + X_t) \qquad (x \in \mathbf{Z}^d) \tag{0.8}$$

is not stationary, and neither are the increments of $X_t$. However, if $w$ is drawn not from $\mu$ but from $\mu_0$ given by

$$\frac{d\mu_0}{d\mu}(w) = \frac{w(0)}{M}, \qquad M = \int w(0)\,\mu(dw) \tag{0.9}$$

then $W_t$ is stationary, ergodic, and even reversible (see Proposition 1 below).

Our first result concerns the correction to the diffusive behavior

$$\mathbf{E}_\mu X_t^2 \sim M^{-1}t \qquad (t \to \infty) \tag{0.10}$$

where $\mathbf{E}_\mu$ denotes the expectation over walk and environment. To see the connection with long-time tails, we establish the relation (see Proposition 3 below)

$$M^{-1}\Delta_t = \mathbf{E}_{\mu_0}([w^{-1}(X_0) - M^{-1}][w^{-1}(X_t) - M^{-1}]) \tag{0.11}$$

with

$$\Delta_t = \frac{d}{dt}\mathbf{E}_\mu(X_t^2 - M^{-1}t)$$

This is the analog of (0.2) and expresses the correction to (0.10) in terms of an equilibrium autocorrelation function. The jump intensity $w^{-1}(X_t) = W_t^{-1}(0)$ is the analog in our model of the velocity $V_t$ in Lorentz models. In Section 3 we prove:

**Theorem 1.** Let $\mu$ be i.i.d. and let $M$ and $V^2$ be the mean and variance of $w(0)$ under $\mu$. Then

$$\lim_{t \to \infty} t^{d/2}\Delta_t = V^2 M^{d/2-3}\left(\frac{d}{2\pi}\right)^{d/2} \tag{0.12}$$

Our second result concerns what happens when $w$ is fixed, i.e., we consider the random variable

$$\Delta_t(w) = \frac{d}{dt}\mathbf{E}_w(X_t^2 - M^{-1}t) \tag{0.13}$$

where $\mathbf{E}_w$ denotes the expectation over the walk given $w$. In Section 4, we prove:

**Theorem 2.** Let $\mu$ be i.i.d. and suppose that:

(i) $\mu(w(0) \geqslant a) = 1$ for some $a > 0$.

(ii) $\int e^{\xi w(0)} \mu(dw) < \infty$ for $\xi$ in a neighborhood of zero.

Then we have

$$\lim_{t \to \infty} t^{d/4} \Delta_{st}(w) = Z_s \quad \text{weakly on} \quad D([\varepsilon, \infty), \mathbf{R}) \quad \text{for all} \quad \varepsilon > 0 \tag{0.14}$$

where $D([\varepsilon, \infty), \mathbf{R})$ is the Skorokhod path space and $\{Z_s : s > 0\}$ is the Gaussian process with

$$\begin{aligned} \mathbf{E} Z_s &= 0 \\ \mathbf{E} Z_r Z_s &= V^2 M^{d/2-4} \left(\frac{d}{2\pi}\right)^{d/2} \frac{1}{(r+s)^{d/2}} \end{aligned} \tag{0.15}$$

The main surprise is that the powers of the long-time tails differ by a factor 2, namely, $t^{-d/4}$ for fixed environment as opposed to $t^{-d/2}$ in averaged environment.[4]

Our paper is organized as follows. In Section 1 we start with definitions and basic relations. In Section 2 we give a heuristic derivation of the long-time tails. In Sections 3 and 4 we give the full proof of Theorems 1 and 2.

## 1. DEFINITIONS AND PRELIMINARIES

In this section we set up the model, introduce the environment process, and establish some basic relations.

### 1.1. Random Walk in Random Environment

Let

$$\Omega = \{w : \mathbf{Z}^d \to (0, \infty)\} \tag{1.1}$$

[4] The same phenomenon was found by van Beijeren[13] for a simple random walk on a random subset of the one-dimensional lattice $\mathbf{Z}$. The additional fluctuations due to the randomness of the medium are called "Sinai fluctuations."

An element of $\Omega$ is called an *environment of waiting times.* A *random environment of waiting times* is a random variable with values in $\Omega$ and distribution $\mu$ satisfying

$$\begin{aligned} &\text{(i)}\quad \mu \text{ is stationary and ergodic under translations} \\ &\text{(ii)}\quad \int w^{-1}(0)\,\mu(dw) < \infty \end{aligned} \tag{1.2}$$

Given $w \in \Omega$, define the operator $L_w$ formally by

$$L_w f(x) = \frac{1}{w(x)} \frac{1}{2d} \sum_{e \in S_d} [f(x+e) - f(x)] \tag{1.3}$$

where $S_d = \{e \in \mathbf{Z}^d\colon |e| = 1\}$ is the set of unit vectors in $\mathbf{Z}^d$. Under asumption (1.2)(ii), $L_w$ is $\mu$-a.s. the generator of a Markov process $\{X_t\colon t \geqslant 0\}$ $(X_0 = 0)$ on $\mathbf{Z}^d$ (ref. 4, p. 815). The latter is called the *random walk in environment* $w$. The interpretation of $L_w$ is the following. From position $x$ the walk jumps at rate $w^{-1}(x)$, i.e., has an exponential waiting time with mean $w(x)$. When the walk jumps it goes to one of the nearest neighbor sites $x + e$ with equal probability $1/2d$.

Let $P_w$ denote the path space measure associated with $L_w$. Define the *random walk in random environment* $\mu$ as the process $\{X_t\colon t \geqslant 0\}$ on $\mathbf{Z}^d$ $(X_0 = 0)$ with path space measure

$$P_\mu = \int P_w\, \mu(dw) \tag{1.4}$$

## 1.2. Simple Random Walk (SRW)

If $w(x) = 1$ for all $x \in \mathbf{Z}^d$, then $L_w$ is the generator of the *simple random walk* with transition probabilities

$$p_t(x, y) = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} p^n(x, y) \tag{1.5}$$

where

$$\begin{aligned} p^0(x, y) &= \delta_{x,y} \\ p^1(x, y) &= 1/2d \qquad \text{if} \quad y = x + e \quad \text{for some} \quad e \in S_d \\ &= 0 \qquad\quad \text{otherwise} \\ p^n(x, y) &= \sum_{z \in \mathbf{Z}^d} p^1(x, z)\, p^{n-1}(z, y) \qquad (n \geqslant 2) \end{aligned} \tag{1.6}$$

Since this process plays an important role in what follows, we give it a separate notation, namely $\{\tilde{X}_t: t \geqslant 0\}$ ($\tilde{X}_0 = 0$). The associated path space measure is denoted by $\tilde{P}$.

### 1.3. Environment Process (EP)

The *environment process* plays an important role in the study of particle systems in random media.[4] In our setting this is the process of environments of waiting times as seen from the position of the walk.

More precisely, for $w \in \Omega$ and $a \in \mathbf{Z}^d$ let $\tau_a w$ be the configuration $w$ shifted by $a$, i.e., $\tau_a w(x) = w(x+a)$. The EP associated with $\{X_t: t \geqslant 0\}$ is defined as the process $\{W_t: t \geqslant 0\}$ on $\Omega$ given by

$$W_t = \tau_{X_t} w \tag{1.7}$$

This has generator

$$Lf(w) = \frac{1}{w(0)} \frac{1}{2d} \sum_{e \in S_d} [f(\tau_e w) - f(w)] \tag{1.8}$$

(ref. 4, p. 817). With a slight abuse of notation we shall use the symbol $P_w$ to denote the path space measure of the EP given $w$.

### 1.4. Independent Environment Process (IEP)

It will turn out to be important to also consider the *independent environment process* $\{\tilde{W}(t): t \geqslant 0\}$ on $\Omega$ given by

$$\tilde{W}_t = \tau_{\tilde{X}_t} w \tag{1.9}$$

i.e., the process of environments of waiting times as seen from an independent SRW (the walk observes $w$ but does not depend on it). This has generator

$$\tilde{L}f(w) = \frac{1}{2d} \sum_{e \in S_d} [f(\tau_e w) - f(w)] \tag{1.10}$$

Note that

$$L = \psi \tilde{L} \tag{1.11}$$

where $\psi: \Omega \to \mathbf{R}$ is

$$\psi(w) = \frac{1}{w(0)} \tag{1.12}$$

The path space measure of the IEP given $w$ is denoted by $\tilde{P}_w$, and $\tilde{P}_\mu = \int \tilde{P}_w \, \mu(dw)$.

### 1.5. Reversible Measures for EP

Assume in addition to (1.2)

$$\int w(0)\, \mu(dw) < \infty \tag{1.13}$$

**Proposition 1.** Let $\mu$ satisfy (1.2). Define $\mu_0$ via

$$\frac{d\mu_0}{d\mu}(w) = \frac{w(0)}{M} \tag{1.14}$$

where $M = \int w(0)\, \mu(dw)$. Then $\mu_0$ is reversible and ergodic for the EP.

*Proof.* See De Masi *et al.*,[4] Lemma 4.3. ▮

The reversibility of $\mu_0$ is equivalent to $L$ being self-adjoint on $L^2(\mu_0)$.

### 1.6. A Feynman–Kac Formula

The following proposition relates expectations for the EP with generator $L$ to expectations for the IEP with generator $\tilde{L}$, and is based on (1.11) and (1.12).

**Proposition 2.** For all $\lambda > 0$, $f \in L^2(\mu_0)$ and $\mu$-a.s. all $w$

$$\int_0^\infty dt\, e^{-\lambda t} \mathbf{E}_w f(W_t)$$

$$= \int_0^\infty dt\, \tilde{\mathbf{E}}_w \left( \exp\left[ -\lambda \int_0^t ds\, \tilde{W}_s(0) \right] \tilde{W}_t(0)\, f(\tilde{W}_t) \right) \tag{1.15}$$

*Proof.* The l.h.s. of (1.15) equals $(\lambda - L)^{-1} f(w)$. Via (1.11)

$$\begin{aligned}(\lambda - L)^{-1} f(w) &= \left( \frac{\lambda}{\psi} - \tilde{L} \right)^{-1} \left( \frac{f}{\psi} \right)(w) \\ &= \int_0^\infty dt \exp\left[ -t \left( \frac{\lambda}{\psi} - \tilde{L} \right) \right] \left( \frac{f}{\psi} \right)(w) \\ &= \int_0^\infty dt\, \tilde{\mathbf{E}}_w \left( \exp\left[ -\int_0^t ds \frac{\lambda}{\psi}(\tilde{W}_s) \right] \left( \frac{f}{\psi} \right)(\tilde{W}_t) \right) \end{aligned} \tag{1.16}$$

where the last equality follows from the Feynman–Kac formula. Substitute (1.12) in (1.16) to obtain (1.15). ▮

## 1.7. Mean Square Displacement

For $x \in \mathbf{Z}^d$ write $x^2 = |x|^2$ with $|\cdot|$ the Euclidean norm.

**Proposition 3.** For all $\mu$ satisfying (1.2):

(i) $X_t^2 - \int_0^t ds\, w^{-1}(X_s)$ is a martingale w.r.t. the canonical filtration $\sigma(X_s\colon 0 \leqslant s \leqslant t)$ for $\mu$-a.s. all $w$.

(ii) $\Delta_t = M\mathbf{E}_{\mu_0}([w^{-1}(0) - M^{-1}][w^{-1}(X_t) - M^{-1}])$.

(iii) $t \to \Delta_t$ is completely monotonic.

(iv) There exists a nondecreasing right-continuous function $\alpha(\gamma)$ on $[0, \infty)$ with $\alpha(0) = 0$ such that

$$\Delta_t = \int_0^\infty e^{-\gamma t}\, d\alpha(\gamma) \tag{1.17}$$

*Proof.* (i) Let $f(x) = x^2$. Fix $w$. Compute

$$(L_w f)(x) = w^{-1}(x)(2d)^{-1} \sum_{e \in S_d} [(x+e)^2 - x^2] = w^{-1}(x)$$

Then use that $f(X_t) - f(X_0) - \int_0^t ds\, L_w f(X_s)$ is a martingale because $\{X_t\colon t \geqslant 0\}$ is Markov with generator $L_w$ (ref. 9, Section I.5).

(ii) From (i) and (1.14) follows

$$\begin{aligned} \mathbf{E}_\mu(X_t^2 - M^{-1}t) &= \mathbf{E}_\mu\left(\int_0^t ds\, [w^{-1}(X_s) - M^{-1}]\right) \\ &= M\mathbf{E}_{\mu_0}\left(\int_0^t ds\, w^{-1}(0)[w^{-1}(X_s) - M^{-1}]\right) \\ &= M\mathbf{E}_{\mu_0}\left(\int_0^t ds\, [w^{-1}(0) - M^{-1}][w^{-1}(X_s) - M^{-1}]\right) \end{aligned} \tag{1.18}$$

where the last equality uses that $\mu_0$ is stationary for the EP (Proposition 1) and $\mathbf{E}_{\mu_0} w^{-1}(0) = M^{-1}$.

(iii) Note that by (ii)

$$M^{-1}\Delta_t = \langle \varphi, S(t)\, \varphi \rangle \tag{1.19}$$

where $\varphi(w) = w^{-1}(0) - M^{-1}$, $\langle \cdot, \cdot \rangle$ is the inner product over $L^2(\mu_0)$, and $\{S(t)\colon t \geqslant 0\}$ is the semigroup of the EP on $L^2(\mu_0)$ [recall (1.7)]. By reversibility of $\mu_0$ (Proposition 1) the semigroup is self-adjoint and $-L$ is self-adjoint and nonnegative. Hence

$$\left(\frac{d}{dt}\right)^n \langle\varphi, S(t)\,\varphi\rangle = \langle\varphi, L^n S(t)\,\varphi\rangle = (-1)^n \|(-L)^{n/2} S(t/2)\,\varphi\|^2 \tag{1.20}$$

(iv) This follows immediately from (1.19) and the spectral theorem for self-adjoint operators (see also ref. 14, Theorem 12 A). Because the EP is ergodic, $-L$ has a simple eigenvalue zero with corresponding eigenspace the constant functions. Since $\langle\varphi, 1\rangle = \mathbf{E}_{\mu_0}\varphi = 0$, it follows that there is no contribution to (1.17) from $\gamma = 0$. ▮

Proposition 3(iv) will be important later when we apply a Tauberian theorem for Stieltjes transforms.

As a final preliminary, we use Proposition 2 to get an identity which relates the Laplace transform of $(d/dt)\,\mathbf{E}_w X_t^2$ to the *local times* of the SRW introduced in Section 1.2. Let

$$l_t(x) = \int_0^t ds\, 1_{\{\tilde{X}_s = x\}} \qquad (t \geqslant 0,\ x \in \mathbf{Z}^d) \tag{1.21}$$

i.e., the amount of time spent at site $x$ up to time $t$ by SRW.

**Proposition 4.** For $\lambda > 0$ and $\mu$-a.s. all $w$

$$\int_0^\infty e^{-\lambda t}\, d\mathbf{E}_w X_t^2 = \int_0^\infty dt\, \tilde{\mathbf{E}}\left(\exp\left[-\lambda \sum_x l_t(x)\, w(x)\right]\right) \tag{1.22}$$

*Proof.* By Proposition 3(i) and Proposition 2 with $f(w) = w^{-1}(0)$ [recall also (1.7)]

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\, d\mathbf{E}_w X_t^2 &= \int_0^\infty dt\, e^{-\lambda t}\mathbf{E}_w(W_t^{-1}(0)) \\ &= \int_0^\infty dt\, \tilde{\mathbf{E}}_w\left(\exp\left[-\lambda\int_0^t ds\, \tilde{W}_s(0)\right]\right) \\ &= \int_0^\infty dt\, \tilde{\mathbf{E}}\left(\exp\left[-\lambda\int_0^t ds\, w(\tilde{X}_s)\right]\right)\end{aligned} \tag{1.23}$$

Note that $\mathbf{E}_{\mu_0} w^{-2}(0) = M^{-1}\mathbf{E}_\mu w^{-1}(0) < \infty$ by (1.2)(ii), hence $f \in L^2(\mu_0)$. Now write

$$\int_0^t ds\, w(\tilde{X}_s) = \sum_x l_t(x)\, w(x) \quad ▮ \tag{1.24}$$

## 2. HEURISTIC DERIVATION OF LONG-TIME TAILS

In this section we give a sketch of how to derive the asymptotic behavior of $(d/dt)\,\mathbf{E}_w X_t^2$ for $t \to \infty$ by analyzing its Laplace transform (1.22) for $\lambda \to 0$. The full proof will be given in Sections 3 and 4. From now on we assume that

$$\begin{aligned} &\text{(i)} \quad \mu \text{ is i.i.d.} \\ &\text{(ii)} \quad \int w^{-1}(0)\,\mu(dw) < \infty \qquad (2.1) \\ &\text{(iii)} \quad \int w^2(0)\,\mu(dw) < \infty \end{aligned}$$

and abbreviate

$$\begin{aligned} M &= \int w(0)\,\mu(dw) \\ V^2 &= \int [w(0) - M]^2\,\mu(dw) \end{aligned} \qquad (2.2)$$

We start from Proposition 4. Using the identity $\sum_x l_t(x) = t$, we make a formal expansion of the r.h.s. of (1.22) for small $\lambda$ as follows:

$$\begin{aligned} &\int_0^\infty e^{-\lambda t}\,d\mathbf{E}_w X_t^2 \\ &\quad = \int_0^\infty dt\,e^{-\lambda M t}\tilde{\mathbf{E}}\left(\exp\left\{-\lambda \sum_x l_t(x)[w(x) - M]\right\}\right) \\ &\quad = \int_0^\infty dt\,e^{-\lambda M t}\left\{1 - \lambda \sum_x \tilde{\mathbf{E}} l_t(x)[w(x) - M]\right. \\ &\qquad \left. + \tfrac{1}{2}\lambda^2 \sum_{x,y} \tilde{\mathbf{E}}(l_t(x)\,l_t(y))[w(x) - M][w(y) - M] - \cdots\right\} \end{aligned} \qquad (2.3)$$

At this point we do not worry about technicalities, such as the smallness of the expansion parameter. These will be handled later. The leading order term in (2.3) is

$$\int_0^\infty dt\,e^{-\lambda M t} = \frac{1}{\lambda M} \qquad (2.4)$$

and reflects

$$\mathbf{E}_w X_t^2 = M^{-1} t + \cdots \qquad (2.5)$$

which corresponds to the well-known result for the diffusion constant

$$\lim_{t\to\infty}\frac{1}{t}\mathbf{E}_w X_t^2 = \lim_{t\to\infty}\frac{1}{t}\int_0^t ds\, \mathbf{E}_w W_s^{-1}(0) = M^{-1} \qquad \mu\text{-a.s.} \tag{2.6}$$

If we subtract the leading order term in (2.3) and interchange the expansion with the integration, then we obtain

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\, d\mathbf{E}_w(X_t^2 - M^{-1}t)\\
&\quad = -\lambda\int_0^\infty dt\, e^{-\lambda M t}\sum_x \tilde{\mathbf{E}} l_t(x)[w(x)-M]\\
&\quad\quad +\tfrac{1}{2}\lambda^2\int_0^\infty dt\, e^{-\lambda M t}\sum_{x,y}\tilde{\mathbf{E}}(l_t(x)\, l_t(y))[w(x)-M][w(y)-M]\\
&\quad\quad -\cdots
\end{aligned} \tag{2.7}$$

We now distinguish two cases:

## 2.1. Fixed Environment

Via the identity $\tilde{\mathbf{E}} l_t(x) = \int_0^t ds\, p_s(0,x)$ [recall (1.5) and (1.21)] the leading order term in (2.7) may be rewritten as

$$\int_0^\infty dt\, e^{-\lambda t}\xi_t(w) \tag{2.8}$$

where

$$\xi_t(w) = -\frac{1}{M^2}\sum_x p_{t/M}(0,x)[w(x)-M] \tag{2.9}$$

This suggests that

$$\Delta_t \sim \xi_t(w) \qquad (t\to\infty) \tag{2.10}$$

To see what the order of magnitude of $\xi_t(w)$ is, let us compute its variance under $\mu$. Using the i.i.d. property of $\mu$ and (2.2), this gives

$$\begin{aligned}
\mathbf{E}_\mu \xi_t^2(w) &= \frac{V^2}{M^4}\sum_x p_{t/M}^2(0,x)\\
&= \frac{V^2}{M^4}\, p_{2t/M}(0,0)\\
&\sim C_1 t^{-d/2} \qquad (t\to\infty)
\end{aligned} \tag{2.11}$$

where $C_1 = C_1(M, V, d) > 0$ is computable via the local limit theorem for SRW. Hence $\xi_t(w)$ is of the order $t^{-d/4}$. Together with the structure of (2.9), this in turn suggests that we have a functional limit theorem

$$\lim_{t \to \infty} t^{d/4} \xi_{st}(w) = Z_s \qquad (s > 0) \tag{2.12}$$

with $\{Z_s : s > 0\}$ some Gaussian process. The combination of (2.10) and (2.12) yields (0.14) in Theorem 2 of the Introduction. In Section 4 we give the proof of the heuristic steps and identify the covariance structure of $\{Z_s : s > 0\}$. To handle the technicalities, it will be necessary to assume conditions (i) and (ii) in Theorem 2.

## 2.2. Averaged Environment

If we average over $w$ in (2.7) under $\mu$, then the first term drops out and the leading order term becomes

$$\tfrac{1}{2}\lambda^2 V^2 \int_0^\infty dt\, e^{-\lambda M t} \sum_x \tilde{\mathbf{E}} l_t^2(x) \tag{2.13}$$

where we use again the i.i.d. property of $\mu$ and (2.2). Via the identity $\sum_x \tilde{\mathbf{E}} l_t^2(x) = 2 \int_0^t ds\, (t-s)\, p_s(0, 0)$, the latter may be rewritten as

$$\int_0^\infty dt\, e^{-\lambda t} \zeta_t \tag{2.14}$$

where

$$\zeta_t = \frac{V^2}{M^3}\, p_{t/M}(0, 0) \tag{2.15}$$

This suggests that

$$\Delta_t \sim \zeta_t \qquad (t \to \infty) \tag{2.16}$$

Since

$$\zeta_t \sim C_2 t^{-d/2} \qquad (t \to \infty) \tag{2.17}$$

$C_2 = C_2(M, V, d) > 0$, (2.16) implies the result of Theorem 1 in the Introduction. In Section 3 we give the complete proof.

In order to make the above calculations rigorous, we have to prove two facts:

A. The $\lambda$-expansion (2.3) makes sense, i.e., $\lambda \sum_x l_t(x)[w(x) - M]$ is small for typical $w$ and for relevant $t$ as $\lambda \to 0$.

B. The asymptotic behavior of $(d/dt)\,\mathbf{E}_w(X_t^2 - M^{-1}t)$ as $t \to \infty$ can be deduced from the $\lambda \to 0$ behavior of the expansion, i.e., a Tauberian argument can be given to conclude (2.10) and (2.16).

## 3. PROOF OF THEOREM 1

### 3.1. Large-Deviation Estimate for Local Times

The following lemmas will be instrumental in settling problem A at the end of Section 2.

**Lemma 1.** For all $\delta \in (0, 1/2)$ there exist $K, \tilde{K} > 0$ such that

$$\tilde{P}(\sup_x l_t(x) > t^{1/2+\delta}) \leqslant \tilde{K} e^{-Kt^{\delta/2}} \qquad \text{for all} \quad t \geqslant 0 \tag{3.1}$$

*Proof.* Fix $\delta \in (0, 1/2)$. Let $\{Y_n : n \in \mathbf{N}\}$ $(Y_0 = 0)$ be SRW in discrete time [with transition probabilities defined in (1.6)]. We use the same symbol $\tilde{P}$ to denote its path space measure. Define its local times

$$L_n(x) = \sum_{m=0}^{n} 1_{\{Y_m = x\}} \qquad (n \in \mathbf{N},\, x \in \mathbf{Z}^d) \tag{3.2}$$

First we prove that there exists $K_1 \in (0, \infty)$ such that

$$\tilde{P}(\sup_x L_n(x) > n^{1/2+\delta/2}) \leqslant (n+1)\, e^{-K_1 n^{\delta}} \qquad \text{for all} \quad n \in \mathbf{N} \tag{3.3}$$

Indeed, let $h(n) = \lfloor n^{1/2+\delta/2} \rfloor$, with $\lfloor \cdot \rfloor$ the greatest integer function. Then the l.h.s. of (3.3) is bounded above by

$$\begin{aligned}
&\tilde{P}(\sup_x L_n(x) > h(n)) \\
&\quad \leqslant \sum_{0 \leqslant i \leqslant n} \tilde{P}(L_n(Y_i) > h(n),\ Y_j \neq Y_i \text{ for } 0 \leqslant j < i) \\
&\quad \leqslant (n+1)\, \tilde{P}(L_n(0) > h(n))
\end{aligned} \tag{3.4}$$

Let $\rho_k$ $(k \geqslant 1)$ denote the time at which $Y_n$ returns to the origin for the $k$th time. By the Markov inequality

$$\begin{aligned}
\tilde{P}(L_n(0) > h(n)) &= \tilde{P}(\rho_{h(n)} \leqslant n) \\
&\leqslant \inf_{\xi > 0} (e^{\xi n} (\tilde{\mathbf{E}} e^{-\xi \rho_1})^{h(n)})
\end{aligned} \tag{3.5}$$

because $\rho_k$ is a sum of $k$ independent copies of $\rho_1$. The following identity is standard (ref. 10, Proposition I.2):

$$\tilde{\mathbf{E}} e^{-\xi \rho_1} = 1 - G^{-1}(e^{-\xi}) \tag{3.6}$$

where

$$G(z)=\sum_{n=0}^{\infty} z^n \tilde{P}(Y_n=0) \tag{3.7}$$

Since for any $d\geqslant 1$ there exists $A_1>0$ such that $\tilde{P}(Y_n=0)\leqslant A_1 n^{-1/2}$ for all $n$ (ref. 10, Proposition 7.6), we have from (3.6) and (3.7)

$$\tilde{\mathbf{E}}e^{-\xi\rho_1}\leqslant e^{-A_2\xi^{1/2}} \qquad \text{for all} \quad \xi>0 \quad \text{and some} \quad A_2>0 \tag{3.8}$$

Now take $\xi=[A_2 h(n)/2n]^2$, the value where the bound (3.5) attains its infimum after substitution of (3.8). This gives

$$\tilde{P}(L_n(0)>h(n))\leqslant \exp[-A_2^2 h^2(n)/4n]$$

Substitution into (3.4) yields (3.3). Next we note that

$$l_t(x)\leqslant \sum_{k=0}^{L_{N_t}(x)} \tau_k(x) \tag{3.9}$$

where $N_t=\max\{j\in \mathbf{N}: \sum_{0\leqslant i\leqslant j}\tau_i\leqslant t\}$ is the number of steps of SRW in continuous time prior to time $t$, $\tau_i$ is the $i$th waiting time, and $\tau_k(x)$ is the $k$th waiting at $x$. Note that $\tau_i$ and the $\tau_k(x)$ are i.i.d. exponential with mean 1 (and that $\{\tau_i: 0\leqslant i\leqslant N_t\}=\{\tau_k(x): x\in \mathbf{Z}^d,\ 0\leqslant k\leqslant L_{N_t}(x)\}$). Hence

$$\begin{aligned}\tilde{P}(\sup_x l_t(x)>t^{1/2+\delta})&\leqslant \tilde{P}(N_t>\lfloor 2t\rfloor)+\tilde{P}(\sup_x L_{\lfloor 2t\rfloor}(x)>\lfloor 2t\rfloor^{1/2+\delta/2})\\&\quad+\tilde{P}(\tau_i>\lfloor \tfrac{1}{2}t^{\delta/2}\rfloor \text{ for some } 0\leqslant i\leqslant \lfloor 2t\rfloor)\\&\leqslant e^{-K_2 t}+e^{-K_3 t^{\delta}}+e^{-K_4 t^{\delta 2}}\end{aligned} \tag{3.10}$$

The second term in the r.h.s. comes from (3.3). The other terms are standard estimates. ∎

The following lemma derives from Lemma 1 and will be important in the sequel. Define the event

$$A_{t,\lambda,\alpha,\delta}=\{\sup_x l_t(x)\leqslant \lambda^{-\alpha(1/2+\delta)}\} \tag{3.11}$$

**Lemma 2.** For $\alpha>0$, $\delta\in(0,1/2)$, and $t\in[0,\lambda^{-\alpha}]$

$$P(A^c_{t,\lambda,\alpha,\delta})\leqslant P(A^c_{\lambda^{-\alpha},\lambda,\alpha,\delta})\leqslant \tilde{K}e^{-K\lambda^{-\alpha\delta 2}} \tag{3.12}$$

*Proof.* Straightforward from Lemma 1. ∎

### 3.2. λ-Expansion for Truncated Laplace Integral

Start from Proposition 4 and average (1.22) over $\mu$ using the i.i.d. property

$$\int_0^\infty e^{-\lambda t}\, d\mathbf{E}_\mu X_t^2 = \int_0^\infty dt\, \tilde{\mathbf{E}}_\mu \left( \exp\left[ -\lambda \sum_x l_t(x)\, w(x) \right] \right)$$
$$= \int_0^\infty dt\, e^{-\lambda M t} \tilde{\mathbf{E}} \left( \prod_x \Gamma(\lambda l_t(x)) \right) \tag{3.13}$$

where

$$\Gamma(\xi) = \int e^{-\xi[w(0)-M]}\, \mu(dw) \qquad (\xi \geqslant 0) \tag{3.14}$$

In Lemma 3 below we show that there exists $\alpha > 1$ such that if the integral in the r.h.s. of (3.13) is restricted to $t \in [0, \lambda^{-\alpha}]$, then it can be expanded for small $\lambda$. In Section 3.3 (Lemmas 4 and 5 below) we show that the remaining part of the integral is negligible.

**Lemma 3.** For $\alpha \in (0, 4/3)$ and $\delta > 0$ sufficiently small

$$\int_0^{\lambda^{-\alpha}} dt\, e^{-\lambda M t}\, \tilde{\mathbf{E}} \left( \prod_x \Gamma(\lambda l_t(x)) \right)$$
$$= \int_0^{\lambda^{-\alpha}} dt\, e^{-\lambda M t} \left\{ 1 + \tfrac{1}{2}\lambda^2 V^2 [1 + o(1)] \sum_x \tilde{\mathbf{E}} l_t^2(x) \right\} + O(e^{-\lambda^{-\alpha\delta/2}}) \tag{3.15}$$

*Proof.* By Lemma 2, and the fact that the integrand in the l.h.s. of (3.15) is at most 1 [recall (3.13)],

$$\int_0^{\lambda^{-\alpha}} dt\, e^{-\lambda M t}\, \tilde{\mathbf{E}} \left( \prod_x \Gamma(\lambda l_t(x)) \right)$$
$$= \int_0^{\lambda^{-\alpha}} dt\, e^{-\lambda M t}\, \tilde{\mathbf{E}} \left( \prod_x \Gamma(\lambda l_t(x)) \,\middle|\, A_{t,\lambda,\alpha,\delta} \right) + O(e^{-\lambda^{-\alpha\delta/2}}) \tag{3.16}$$

On the event $A_{t,\lambda,\alpha,\delta}$

$$\lambda l_t(x) \leqslant \lambda^{1-\alpha(1/2+\delta)} \qquad \text{for all} \quad x \quad \text{and} \quad t \in [0, \lambda^{-\alpha}] \tag{3.17}$$

Hence, if

$$1 - \alpha(\tfrac{1}{2} + \delta) > 0 \tag{3.18}$$

then we can expand $\Gamma$ in (3.14) using (2.2) to obtain

$$\Gamma(\lambda l_t(x)) = 1 + \tfrac{1}{2}\lambda^2 V^2 l_t^2(x)[1 + o(1)] \qquad (\lambda \to 0) \tag{3.19}$$

where the $o(1)$ tends to zero as $\lambda \to 0$ uniformly in $x$ and $t \in [0, \lambda^{-\alpha}]$. It follows that

$$\tilde{\mathbf{E}}\left(\prod_x \Gamma(\lambda l_t(x)) \middle| A_{t,\lambda,\alpha,\delta}\right)$$
$$= \tilde{\mathbf{E}}\left(\exp\left\{\tfrac{1}{2}\lambda^2 V^2[1 + o(1)] \sum_x l_t^2(x)\right\} \middle| A_{t,\lambda,\alpha,\delta}\right) \tag{3.20}$$

Next use $\sum_x l_t(x) = t$ to note that on the event $A_{t,\lambda,\alpha,\delta}$ with $t \in [0, \lambda^{-\alpha}]$

$$\lambda^2 \sum_x l_t^2(x) \leqslant \lambda^2 t \sup_x l_t(x) \leqslant \lambda^{2-\alpha(3/2+\delta)} \tag{3.21}$$

Hence, if

$$2 - \alpha(\tfrac{3}{2} + \delta) > 0 \tag{3.22}$$

then we can expand the exponential in the r.h.s. of (3.20) to prove the claim. The restrictions (3.18) and (3.22) can be met by picking $\alpha \in (0, 4/3)$ and $\delta > 0$ sufficiently small. ∎

### 3.3. Remaining Part of Laplace Integral

For reasons that will become clear along the way, we split the remaining integration interval $(\lambda^{-\alpha}, \infty)$ into two parts, namely $(\lambda^{-\alpha}, \lambda^{-\beta}]$ and $(\lambda^{-\beta}, \infty)$, where $\beta > 2$. In Lemmas 4 and 5 below we show that both parts have a negligible contribution to the integral of the r.h.s. of (3.13) as $\lambda \to 0$.

**Lemma 4.** For $\alpha \in (1, 4/3)$, $\beta > \alpha$, and $\delta > 0$ sufficiently small

$$\int_{\lambda^{-\alpha}}^{\lambda^{-\beta}} dt\, e^{-\lambda M t} \tilde{\mathbf{E}}\left(\prod_x \Gamma(\lambda l_t(x))\right) = O((\lambda^{-\beta} - \lambda^{-\alpha})\, e^{-\lambda^{-\alpha\delta/2}}) \tag{3.23}$$

*Proof.* First note that the integrand in (3.23) is decreasing in $t$ [recall (3.13)]. Therefore the integral can be bounded above by $\lambda^{-\beta} - \lambda^{-\alpha}$

times the value of the integrand at $t=\lambda^{-\alpha}$. Next note that at $t=\lambda^{-\alpha}$ we can use Lemma 2 and (3.20)–(3.22) to estimate

$$\begin{aligned} &e^{-\lambda M t}\tilde{\mathbf{E}}\left(\prod_x \Gamma(\lambda l_t(x))\right) \\ &\quad\leqslant e^{-\lambda M t}\tilde{\mathbf{E}}\left(\exp\left\{\tfrac{1}{2}\lambda^2 V^2[1+o(1)]\sum_x l_t^2(x)\right\}\Bigg| A_{t,\lambda,\alpha,\delta}\right)+\tilde{P}(A^c_{t,\lambda,\alpha,\delta}) \\ &\quad= e^{-M\lambda^{1-\alpha}}[1+o(1)]+O(e^{-\lambda^{-\alpha\delta/2}}) \quad \blacksquare \end{aligned} \tag{3.24}$$

**Lemma 5.** For $\beta>2$

$$\int_{\lambda^{-\beta}}^{\infty} dt\, e^{-\lambda M t}\tilde{\mathbf{E}}\left(\prod_x \Gamma(\lambda l_t(x))\right)=O(e^{-\lambda^{-(\beta-2)/3}}) \tag{3.25}$$

*Proof.* Return to (3.2) and (3.9). First note that

$$l_t(x)\geqslant \sum_{k=0}^{L_{N_t}(x)-1}\tau_k(x) \tag{3.26}$$

Hence

$$\begin{aligned} &e^{-\lambda M t}\tilde{\mathbf{E}}\left(\prod_x \Gamma(\lambda l_t(x))\right) \\ &\quad=\tilde{\mathbf{E}}_\mu\left(\exp\left[-\lambda\sum_x l_t(x)\,w(x)\right]\Bigg| N_t\geqslant\lfloor\tfrac{1}{2}t\rfloor\right)+O(e^{-t}) \\ &\quad\leqslant\tilde{\mathbf{E}}_\mu\left(\exp\left[-\lambda\sum_x\sum_{k=0}^{L_{\lfloor t/2\rfloor}(x)-1}\tau_k(x)\,w(x)\right]\right)+O(e^{-t}) \\ &\quad=\tilde{\mathbf{E}}\left(\prod_x I(\lambda, L_{\lfloor t/2\rfloor}(x))\right)+O(e^{-t}) \end{aligned} \tag{3.27}$$

where

$$I(\lambda,l)=\int\left(\frac{1}{1+\lambda w(0)}\right)^l\mu(dw)\qquad (l\in\mathbf{N}) \tag{3.28}$$

In (3.27) we use that the $w(x)$ are i.i.d. under $\mu$ and that the $\tau_k(x)$ are i.i.d. exponential with mean 1 independent of the $w(x)$.

Next, for $\varepsilon>0$ estimate

$$I(\lambda,l)\leqslant\mu(w(0)<\varepsilon)+\mu(w(0)\geqslant\varepsilon)\left(\frac{1}{1+\lambda\varepsilon}\right)^{1\{l>0\}} \tag{3.29}$$

Since $\mu(w(0) < \varepsilon) \to 0$ as $\varepsilon \to 0$, it follows that for $\varepsilon$ small enough (and all $\lambda$ with $\lambda \leqslant 1$)

$$I(\lambda, l) \leqslant e^{-(1/2)\,\lambda\varepsilon 1_{\{l>0\}}} \tag{3.30}$$

Hence

$$\prod_x I(\lambda, L_{\lfloor t/2 \rfloor}(x)) \leqslant e^{-(1/2)\,\lambda\varepsilon R_{\lfloor t/2 \rfloor}} \tag{3.31}$$

with

$$R_n = \sum_x 1_{\{L_n(x) > 0\}} \tag{3.32}$$

the range of discrete-time SRW after $n$ steps. Since

$$n + 1 = \sum_x L_n(x) \leqslant R_n \sup_x L_n(x) \tag{3.33}$$

it follows from (3.3) that

$$\tilde{P}(R_n < n^{1/2-\delta/2}) \leqslant (n+1)\, e^{-K_1 n^{\delta}} \tag{3.34}$$

Finally, by combining (3.27), (3.31), and (3.34), we get

$$\begin{aligned} &\exp(-\lambda M t)\, \tilde{\mathbf{E}}\left(\prod_x \Gamma(\lambda l_t(x))\right) \\ &\quad \leqslant O(e^{-t}) + \exp(-\tfrac{1}{2}\lambda\varepsilon \lfloor \tfrac{1}{2}t \rfloor^{1/2-\delta/2}) \\ &\qquad + (\lfloor \tfrac{1}{2}t \rfloor + 1) \exp(-K_1 \lfloor \tfrac{1}{2}t \rfloor^{\delta}) \end{aligned} \tag{3.35}$$

If we now pick $\delta = (\beta - 2)/3\beta$ and integrate (3.35) over $t \in (\lambda^{-\beta}, \infty)$, then the claim follows. ∎

## 3.4. Spectral Representation and Tauberian Theorem

At this point we have solved problem A at the end of Section 2 for the case of averaged environment. That is to say, by combining (3.13) with Lemmas 3–5 we have proved

$$\begin{aligned} &\int_0^{\infty} e^{-\lambda t}\, d\mathbf{E}_{\mu}(X_t^2 - M^{-1}t) \\ &\qquad = \tfrac{1}{2}\lambda^2 V^2 [1 + o(1)] \int_0^{\infty} dt\, e^{-\lambda M t} \sum_x \tilde{\mathbf{E}} l_t^2(x) \end{aligned} \tag{3.36}$$

which corresponds to (2.13). To be able to conclude (2.16), we must also solve problem B. In other words, from the asymptotic behavior of the r.h.s. of (3.36) as $\lambda \to 0$, computable via (2.14) and (2.15), we want to prove (2.16) and (2.17) via a Tauberian theorem.

The application of Tauberian arguments usually requires some kind of regularity. In our case this regularity comes from Proposition 3(iv), which allows us to proceed as follows. Abbreviate the l.h.s. of (3.36) as

$$H(\lambda) = \int_0^\infty e^{-\lambda t}\, d\mathbf{E}_\mu(X_t^2 - M^{-1}t) \tag{3.37}$$

Substitution of (1.17) gives

$$H(\lambda) = \int_0^\infty \frac{1}{\lambda + \gamma}\, d\alpha(\gamma) \tag{3.38}$$

i.e., $H(\lambda)$ is the Stieltjes transform of the positive measure $d\alpha(\gamma)$. The main step in finishing the proof of Theorem 1 is the following proposition, which identifies $\alpha(\gamma)$ for $\gamma \to 0$.

**Proposition 5.** In $d \geqslant 1$

$$\alpha(\gamma) \sim V^2 M^{d/2-3} \left(\frac{d}{2\pi}\right)^{d/2} \frac{1}{\Gamma(d/2+1)} \gamma^{d/2} \qquad (\gamma \to 0) \tag{3.39}$$

The proof is given in Section 3.5 below. Substitution of (3.39) into (1.17) yields (0.12) in Theorem 1 via a standard Abelian theorem (ref. 14, Theorem VIII.2.1).

For the proof of Proposition 5 we need the following Tauberian theorem for Stieltjes transforms.

**Tauberian Theorem.** Let $m > 0$ and let $\alpha(\gamma)$ be a nondecreasing right-continuous function on $[0, \infty)$ with $\alpha(0) = 0$. Assume that

$$f(\lambda) = \int_0^\infty \frac{1}{(\lambda + \gamma)^m}\, d\alpha(\gamma) \tag{3.40}$$

converges for $\lambda > 0$. Then for any $A \geqslant 0$ and $0 \leqslant \kappa < m$ the following are equivalent:

$$\begin{aligned} &\alpha(\gamma) \sim A \frac{\Gamma(m)}{\Gamma(\kappa+1)\,\Gamma(m-\kappa)} \gamma^\kappa \qquad (\gamma \to 0) \\ &f(\lambda) \sim A \lambda^{\kappa - m} \qquad (\lambda \to 0) \end{aligned} \tag{3.41}$$

*Proof.* See ref. 1, Theorem I.7.4. ∎

### 3.5. Proof of Proposition 5

The r.h.s. of (3.36) may be rewritten as in (2.13)–(2.15). This gives

$$H(\lambda)=\frac{V^2}{M^2}[1+o(1)]\,G(\lambda M) \qquad (\lambda\to 0) \tag{3.42}$$

where

$$G(\lambda)=\int_0^\infty dt\,e^{-\lambda t}p_t(0,0) \qquad (\lambda>0) \tag{3.43}$$

In $d=1$ the local limit theorem for SRW (ref. 10, Proposition 7.9) reads

$$p_t(0,0)\sim\frac{1}{(2\pi t)^{1/2}} \qquad (t\to\infty) \tag{3.44}$$

and hence by the standard Abelian theorem

$$G(\lambda)\sim\left(\frac{1}{2\pi}\right)^{1/2}\Gamma\left(\frac{1}{2}\right)\lambda^{-1/2} \qquad (\lambda\to 0) \tag{3.45}$$

By combining (3.38), (3.42), and (3.45) with the Tauberian theorem for $m=1$, $\kappa=1/2$, and $A=V^2M^{-5/2}(1/2\pi)^{1/2}\,\Gamma(1/2)$, we find (3.39) in $d=1$.

In $d\geqslant 2$ we cannot apply the Tauberian theorem directly because $G(\lambda)$ does not have a power law singularity. However, by the local limit theorem for SRW such a singularity occurs in the higher derivatives

$$\left(-\frac{d}{d\lambda}\right)^p G(\lambda)\sim\left(\frac{d}{2\pi}\right)^{d/2}\Gamma\left(p+1-\frac{d}{2}\right)\lambda^{d/2-1-p} \qquad \left(p>\frac{d}{2}-1,\,\lambda\to 0\right) \tag{3.46}$$

Therefore, recalling (3.42), we have to consider $(d/d\lambda)^p\,H(\lambda)$ with $p>d/2-1$. The rest of the proof uses the following two facts:

$$\left(\frac{d}{d\lambda}\right)^p H(\lambda)\sim\frac{V^2}{M^2}[1+o(1)]\left(\frac{d}{d\lambda}\right)^p G(\lambda M) \qquad (p\geqslant 0,\,\lambda\to 0) \tag{3.47}$$

$$\left(-\frac{d}{d\lambda}\right)^p H(\lambda)\sim\Gamma(p+1)\int_0^\infty\frac{1}{(\lambda+s)^{p+1}}\,d\alpha(s) \qquad (p\geqslant 0,\,\lambda>0) \tag{3.48}$$

The second is immediate from (3.38). The first follows from (3.42) and the observation that the differentiation w.r.t. $\lambda$ may be interchanged with the $\lambda$-expansion in (2.3). This technical point is not entirely trivial, but can be checked by a straightforward application of the exponential estimates in Lemmas 3–5. We refer the reader to the proof of Lemma 10 in Section 4.4 for more details.

By combining (3.46)–(3.48) with the Tauberian theorem for $m = p+1 > d/2$, $\kappa = d/2$, and

$$A = V^2 M^{d/2-3} \left(\frac{d}{2\pi}\right)^{d/2} \Gamma\left(p+1-\frac{d}{2}\right) \Big/ \Gamma(p+1)$$

we find (3.39).

## 4. PROOF OF THEOREM 2

### 4.1. Two Main Propositions

For fixed environment $w$ our quantity of interest

$$\Delta_t(w) = \frac{d}{dt} \mathbf{E}_w(X_t^2 - M^{-1}t) \tag{4.1}$$

is no longer a completely monotonic function of $t$. It therefore has no spectral representation as in (1.17), and consequently its Laplace transform cannot be written as the Stieltjes transform of a positive measure as in (3.38). This implies that we cannot use the Tauberian theorem, which played such a crucial role in solving problem B at the end of Section 2 for the averaged environment.

The way out is to consider complex $\lambda$ and to do the Laplace inversion by hand for $t \to \infty$. The computations involve some technical estimates, for which we need the following restrictions on $\mu$ supplementary to (2.1):

$$\begin{aligned} &\text{(i)} \quad \mu(w(0) \geqslant a) = 1 \qquad \text{for some} \quad a > 0 \\ &\text{(ii)} \quad \int e^{\xi w(0)}\, \mu(dw) < \infty \text{ for } \xi \text{ in a neighborhood of } 0 \end{aligned} \tag{4.2}$$

Recall the random variable $\xi_t(w)$ defined in (2.9),

$$\xi_t(w) = -\frac{1}{M^2} \sum_x p_{t/M}(0, x)[w(x) - M] \tag{4.3}$$

The main obstacle in proving Theorem 2 is to prove (2.10), or, more precisely:

**Proposition 6.** In $d \geqslant 1$

$$\lim_{t \to \infty} t^{d/4}[\Delta_t(w) - \xi_t(w)] = 0 \qquad \text{in } \mu\text{-probability} \tag{4.4}$$

The proof is given in Sections 4.2–4.5.

Once we have Proposition 6 it remains to prove the functional central limit theorem for $\xi_t(w)$. This is a rather easy task, because the $p_{t/M}(0, x)$ satisfy the local limit theorem for SRW and the $w(x)$ are i.i.d. with finite second moment. The result is:

**Proposition 7.** In $d \geqslant 1$

$$\lim_{t \to \infty} t^{d/4} \xi_{st}(w) = Z_s \qquad \text{weakly on} \quad D([\varepsilon, \infty), \mathbf{R}) \quad \text{for all} \quad \varepsilon > 0 \tag{4.5}$$

where $D([\varepsilon, \infty), \mathbf{R})$ is the Skorokhod path space and $\{Z_s : s > 0\}$ is the Gaussian process with

$$\begin{aligned} \mathbf{E} Z_s &= 0 \\ \mathbf{E} Z_r Z_s &= V^2 M^{d/2-4} \left(\frac{d}{2\pi}\right)^{d/2} \frac{1}{(r+s)^{d/2}} \end{aligned} \tag{4.6}$$

The proof is given in Section 4.6. Propositions 6 and 7 yield Theorem 2.

## 4.2. Laplace Inversion

Define

$$\begin{aligned} H(\lambda, w) &= \int_0^\infty dt\, e^{-\lambda t} \Delta_t(w) \\ G(\lambda, w) &= \int_0^\infty dt\, e^{-\lambda t} \xi_t(w) \end{aligned} \tag{4.7}$$

From the inversion formula for the Laplace transform we have that for any $t \geqslant 0$ and $p \geqslant 0$

$$t^p[\Delta_t(w) - \xi_t(w)] = \int_{C_x} d\lambda\, e^{\lambda t} \left(-\frac{d}{d\lambda}\right)^p [H(\lambda, w) - G(\lambda, w)] \tag{4.8}$$

where $C_x = \{\lambda = x + iy : y \in \mathbf{R}\}$ $(x > 0)$ is any vertical line in the complex right half-plane $C_+$. We shall pick $x = t^{-1}$ and show the following:

**Lemma 6.** For $p$ sufficiently large

$$\lim_{t\to\infty} t^{d/4-p} \int_{C_{t^{-1}}} d\lambda\, e^{\lambda t} \left(-\frac{d}{d\lambda}\right)^p [H(\lambda, w) - G(\lambda, w)] = 0 \qquad \text{in } \mu\text{-probability} \tag{4.9}$$

Equations (4.8) and (4.9) yield (4.4) in Proposition 6.

The starting point of our computation is the representation in (1.22), which still holds for $\lambda \in C_+$ by analytic continuation. The idea will be to split $C_{t^{-1}}$ into two parts, namely

$$C^{\gamma}_{t^{-1}} = \{\lambda = t^{-1} + iy \colon |y| \leqslant t^{-\gamma}\} \qquad (\gamma \in (0, 1)) \tag{4.10}$$

and $C_{t^{-1}} \backslash C^{\gamma}_{t^{-1}}$, and to show that (4.9) holds for each part separately. In Section 4.4 we deal with $C^{\gamma}_{t^{-1}}$ using truncation and $\lambda$-expansion of the Laplace integrals in (4.7) analogous to Sections 3.2 and 3.3 (Lemmas 8–11 below). In Section 4.5 we deal with $C_{t^{-1}} \backslash C^{\gamma}_{t^{-1}}$ using certain resolvent estimates for the EP and the IEP (Lemma 12 below).

In Section 4.3 we first prove a large-deviation lemma (Lemma 7 below) needed for the $\lambda$-expansion, i.e., problem A at the end of Section 2.

## 4.3. Large-Deviation Estimate for $\lambda \sum_x l_t(x)[w(x) - M]$

Abbreviate

$$U_t = \sum_x l_t(x)[w(x) - M] \tag{4.11}$$

From (4.3) and $\tilde{\mathbf{E}} l_t(x) = \int_0^t ds\, p_s(0, x)$ follows

$$\frac{d}{dt} \tilde{\mathbf{E}} U_t = -M^2 \xi_{Mt}(w) \tag{4.12}$$

Consider the event $A_{t,|\lambda|,\alpha,\delta}$ defined in (3.11) with $\lambda$ replaced by $|\lambda|$. Lemma 2 continues to hold with this replacement.

**Lemma 7.** For $\alpha \in (0, 4/3)$ and $\delta, \varepsilon > 0$ sufficiently small

$$\sup_{0 \leqslant t \leqslant |\lambda|^{-\alpha}} \tilde{P}_\mu\left(|\lambda U_t| > |\lambda|^\varepsilon \,\middle|\, A_{t,|\lambda|,\alpha,\delta}\right) = O(e^{-|\lambda|^{-\varepsilon}}) \qquad (\lambda \to 0) \tag{4.13}$$

*Proof.* Fix $t \in [0, |\lambda|^{-\alpha}]$. We first prove the estimate in the upper direction. By the Markov inequality and by the i.i.d. property of $\mu$

$$\tilde{P}_\mu\left(|\lambda|\, U_t > |\lambda|^\varepsilon \,\middle|\, A_{t,|\lambda|,\alpha,\delta}\right)$$
$$\leqslant e^{-|\lambda|^{-\varepsilon}} \tilde{\mathbf{E}}\left(\prod_x \Theta(|\lambda|^{1-2\varepsilon} l_t(x)) \,\middle|\, A_{t,|\lambda|,\alpha,\delta}\right) \tag{4.14}$$

where

$$\Theta(\xi) = \int e^{\xi[w(0) - M]} \mu(dw) \tag{4.15}$$

which exists for $\xi$ sufficiently small by (4.2)(ii). On the event $A_{t,|\lambda|,\alpha,\delta}$

$$|\lambda|^{1-2\varepsilon} l_t(x) \leqslant |\lambda|^{1-2\varepsilon-\alpha(1/2+\delta)} \qquad \text{for all} \quad x \tag{4.16}$$

and so if

$$1 - 2\varepsilon - \alpha(\tfrac{1}{2} + \delta) > 0 \tag{4.17}$$

then we can expand in (4.15)

$$\begin{aligned}\Theta(|\lambda|^{1-2\varepsilon} l_t(x)) &= 1 + \tfrac{1}{2}V^2 |\lambda|^{2-4\varepsilon} l_t^2(x)[1 + o(1)] \\ &= \exp\{\tfrac{1}{2}V^2 |\lambda|^{2-4\varepsilon} l_t^2(x)[1 + o(1)]\}\end{aligned} \tag{4.18}$$

where the $o(1)$ tends to zero as $|\lambda| \to 0$ uniformly in $x$ and in $t \in [0, |\lambda|^{-\alpha}]$. Substitution into (4.14) gives

$$\begin{aligned}&\tilde{P}_\mu\left(|\lambda|\, U_t > |\lambda|^\varepsilon \,\middle|\, A_{t,|\lambda|,\alpha,\delta}\right) \\ &\quad \leqslant e^{-|\lambda|^{-\varepsilon}} \tilde{\mathbf{E}}\left(\exp\left\{\tfrac{1}{2}V^2 |\lambda|^{2-4\varepsilon}[1 + o(1)] \sum_x l_t^2(x)\right\} \,\middle|\, A_{t,|\lambda|,\alpha,\delta}\right)\end{aligned} \tag{4.19}$$

Now note that on the event $A_{t,|\lambda|,\alpha,\delta}$, because $\sum_x l_t(x) = t$, and $t \in [0, |\lambda|^{-\alpha}]$,

$$\sum_x l_t^2(x) \leqslant t \sup_x l_t(x) \leqslant |\lambda|^{-\alpha(3/2+\delta)} \tag{4.20}$$

It follows that if

$$2 - 4\varepsilon - \alpha(\tfrac{3}{2} + \delta) > -\varepsilon \tag{4.21}$$

then

$$\tilde{P}_\mu\left(|\lambda|\, U_t > |\lambda|^\varepsilon \,\middle|\, A_{t,|\lambda|,\alpha,\delta}\right) = O(e^{-|\lambda|^{-\varepsilon}}) \tag{4.22}$$

The restrictions (4.17) and (4.21) can be met by picking $\alpha \in (0, 4/3)$ and $\delta, \varepsilon > 0$ sufficiently small.

This proves the estimate in (4.13) in the upper direction. The lower direction is analogous [change from $w(x) - M$ to $M - w(x)$]. ▮

## 4.4. Estimates for $C^{\gamma}_{t^{-1}}$

In this section we prove (4.9) in Lemma 6 for the integral restricted to $\lambda \in C^{\gamma}_{t^{-1}}$ (Lemmas 8–11 below). Lemmas 2 and 7 will play a crucial role in handling problem A, i.e., in showing that the $\lambda$-expansion is valid for the integral in (2.3) after truncation. Our starting point is the following relation, which follows from (1.22), (4.1), (4.7), and (4.11):

$$\frac{1}{\lambda M} + H(\lambda, w) = \int_0^{\infty} ds\, e^{-\lambda M s} \tilde{\mathbf{E}} e^{-\lambda U_s} \tag{4.23}$$

Define the event

$$B_{t,|\lambda|,\varepsilon} = \{ |\lambda U_t| \leqslant |\lambda|^{\varepsilon} \} \tag{4.24}$$

**Lemma 8.** For $\alpha \in (1, 4/3)$, $\gamma \in (1/\alpha, 1)$, $\varepsilon > 0$ sufficiently small, and $\lambda \in C^{\gamma}_{t^{-1}}$

$$\begin{aligned} &\int_0^{|\lambda|^{-\alpha}} ds\, e^{-\lambda M s} \tilde{\mathbf{E}} e^{-\lambda U_s} \\ &= \int_0^{|\lambda|^{-\alpha}} ds\, e^{-\lambda M s} \tilde{\mathbf{E}}((e^{-\lambda U_s})\, 1_{B_{s,|\lambda|,\varepsilon}}) + O_{\mu}(e^{-|\lambda|^{-\varepsilon}}) \\ &= \frac{1}{\lambda M} + G(\lambda, w) + [1 + o_{\mu}(1)] \int_0^{\infty} ds\, e^{-\lambda M s} \frac{1}{2} \lambda^2 \tilde{\mathbf{E}} U_s^2 \\ &\quad + O_{\mu}(e^{-|\lambda|^{-\varepsilon}}) + O_{\mu}(e^{-t^{\alpha\gamma - 1}}) \qquad (t \to \infty) \end{aligned} \tag{4.25}$$

*Proof.* The first equality follows by combining Lemmas 2 and 7, together with the fact that the integrand has modulus $\leqslant 1$ (pick $\varepsilon < \frac{1}{2}\alpha\delta$ in Lemma 2). The second equality follows by expanding the exponential up to order $\lambda^2$ and afterward extending the integral to $s \in [0, \infty)$. The latter introduces an additional error with modulus of order $\exp(-t^{-1} M |\lambda|^{-\alpha}) = O(\exp(-t^{\alpha\gamma - 1}))$, because $\operatorname{Re} \lambda = t^{-1}$ and $|\lambda| \sim t^{-\gamma}$ $(t \to \infty)$ on $C^{\gamma}_{t^{-1}}$. Use that via (4.7) and (4.12)

$$G(\lambda, w) = -\int_0^{\infty} ds\, e^{-\lambda M s} \lambda \tilde{\mathbf{E}} U_s \tag{4.26}$$

which explains why this term appears as the first-order term in the expansion. ∎

**Lemma 9.** For $\alpha > 1$, $\gamma \in (1/\alpha, 1)$, and $p \geqslant 0$

$$\begin{aligned} &\sup_{\lambda \in C^{\gamma}_{t^{-1}}} \left| \int_{|\lambda|^{-\alpha}}^{\infty} ds \left( -\frac{d}{d\lambda} \right)^p (e^{-\lambda M s} \tilde{\mathbf{E}} e^{-\lambda U_s}) \right| \\ &\quad = O_{\mu}(e^{-t^{\alpha\gamma - 1}}) \qquad (t \to \infty) \end{aligned} \tag{4.27}$$

*Proof.* Abbreviate

$$V_s = \sum_x l_s(x)\, w(x) = U_s + Ms \tag{4.28}$$

Fix $\lambda \in C^{\gamma}_{t^{-1}}$.

First consider $p = 0$. Since $\mathrm{Re}\, \lambda = t^{-1}$, $w(x) \geqslant a$ for all $x$ [by (4.2)(i)], and $\sum_x l_s(x) = s$, it follows that

$$\left| \int_{|\lambda|^{-\alpha}}^{\infty} ds\, \tilde{\mathbf{E}} e^{-\lambda V_s} \right| \leqslant \int_{|\lambda|^{-\alpha}}^{\infty} ds\, e^{-t^{-1}as} \tag{4.29}$$

Since $|\lambda| \leqslant (t^{-2} + t^{-2\gamma})^{1/2} \leqslant 2t^{-\gamma}$ for $t$ sufficiently large, the r.h.s. of (4.29) is bounded above by the same integral over $s \in [2^{-\alpha} t^{\alpha\gamma}, \infty)$. This proves the claim.

Next consider $p \geqslant 1$. For fixed $s$ estimate

$$\begin{aligned} \int \left| \left( -\frac{d}{d\lambda} \right)^p \tilde{\mathbf{E}} e^{-\lambda V_s} \right| \mu(dw) &= \int |\tilde{\mathbf{E}}((V_s)^p\, e^{-\lambda V_s})|\, \mu(dw) \\ &\leqslant \int [\tilde{\mathbf{E}}(V_s)^{2p}]^{1/2}\, (\tilde{\mathbf{E}}\, |e^{-\lambda V_s}|^2)^{1/2}\, \mu(dw) \\ &\leqslant C_p s^p e^{-t^{-1}as} \end{aligned} \tag{4.30}$$

In the first step we interchange the $p$-fold differentiation with the expectation, which is allowed by dominated convergence. In the second step we use Cauchy–Schwarz. In the final step, we use that $w(x) \geqslant a$ for all $x$ and that all moments of $w(x)$ are finite [recall (4.2)]. From (4.30) after integration over $s \in [|\lambda|^{-\alpha}, \infty)$ the claim follows. ▮

**Lemma 10.** For all $p \geqslant 0$ and $\lambda \in C^{\gamma}_{t^{-1}}$

$$\begin{aligned} &\left( -\frac{d}{d\lambda} \right)^p [H(\lambda, w) - G(\lambda, w)] \\ &\qquad = [1 + o_\mu(1)] \left( -\frac{d}{d\lambda} \right)^p \left( \int_0^{\infty} ds\, e^{-\lambda M s} \frac{1}{2} \lambda^2 \tilde{\mathbf{E}} U_s^2 \right) \end{aligned} \tag{4.31}$$

*Proof.* For $p = 0$ the claim follows by combining (4.23) with Lemmas 8 and 9. To get $p \geqslant 1$, first use (4.23) to write [recall (4.28)]

$$\begin{aligned} \left( -\frac{d}{d\lambda} \right)^p \left[ \frac{1}{\lambda M} + H(\lambda, w) \right] &= \int_0^{\infty} ds \left( -\frac{d}{d\lambda} \right)^p \tilde{\mathbf{E}} e^{-\lambda V_s} \\ &= \int_0^{\infty} ds\, \lambda^{-p} \tilde{\mathbf{E}} f_p(\lambda V_s) \end{aligned} \tag{4.32}$$

where $f_p(x) = x^p e^{-x}$. As before, split the integration interval in the r.h.s. of (4.32) into two parts, namely $[0, |\lambda|^{-\alpha}]$ and $(|\lambda|^{-\alpha}, \infty)$, where $\alpha \in (1, 4/3)$. The integral over $(|\lambda|^{-\alpha}, \infty)$ is of $O_\mu(\exp(-t^{\alpha\gamma-1}))$ by Lemma 9 and so can be absorbed in the $o_\mu(1)$ term in (4.31). Recalling Lemmas 2, 7, and 8, we can insert the indicator of the event $B_{s,|\lambda|,\varepsilon}$ in (4.24) under the integral over $[0, |\lambda|^{-\alpha}]$, making an error of $O_\mu(\exp(-|\lambda|^{-\varepsilon})) + O_\mu(\exp(-t^{\alpha\gamma-1}))$. After that we can expand the function $f_p$ around $x = \lambda M s$, recalling (4.28). It is easy to see that by this procedure we obtain an expansion of the l.h.s. of (4.32) which exactly coincides with the expansion that we would have obtained had we simply interchanged $(-d/d\lambda)^p$ and the $\lambda$-expansion in (2.3). Therefore we indeed conclude (4.31). ∎

**Lemma 11.** For $d \geq 1$, $\gamma > 1 - d/12$, and $p$ sufficiently large

$$\lim_{t\to\infty} t^{d/4-p} \left| \int_{C^\gamma_{t^{-1}}} d\lambda\, e^{\lambda t} \left[ \left(-\frac{d}{d\lambda}\right)^p \int_0^\infty ds\, e^{-\lambda M s} \frac{1}{2} \lambda^2 \tilde{\mathbf{E}} U_s^2 \right] \right| = 0$$

in $\mu$-probability (4.33)

*Proof.* First compute

$$\left(-\frac{d}{d\lambda}\right)^p \lambda^2 e^{-\lambda M s} = (Ms)^{p-2} e^{-\lambda M s}[(\lambda M s)^2 - 2p\lambda M s + p(p-1)] \tag{4.34}$$

Since $\lambda \in C^\gamma_{t^{-1}}$, we have for $t$ large enough

$$\left| \left(-\frac{d}{d\lambda}\right)^p \lambda^2 e^{-\lambda M s} \right| \leq C_p (s^p t^{-2\gamma} + s^{p-1} t^{-\gamma} + s^{p-2})\, e^{-t^{-1} M s} \tag{4.35}$$

Next use the identity

$$\mathbf{E}_\mu \tilde{\mathbf{E}} U_s^2 = V^2 \sum_x \tilde{\mathbf{E}} l_s^2(x) = 2V^2 \int_0^s dr\, (s-r)\, p_r(0,0)\, dr$$

to estimate

$$\begin{aligned} &\int \mu(dw) \left| \left(-\frac{d}{d\lambda}\right)^p \int_0^\infty ds\, e^{-\lambda M s} \frac{1}{2} \lambda^2 \tilde{\mathbf{E}} U_s^2 \right| \\ &\quad \leq 2C_p V^2 \int_0^\infty ds\, (s^p t^{-2\gamma} + s^{p-1} t^{-\gamma} + s^{p-2}) \\ &\qquad \times e^{-t^{-1} M s} \int_0^s dr\, (s-r)\, p_r(0,0) \\ &\quad = O(t^{3+p-d/2-2\gamma}) \qquad \text{if} \quad 3 + p - \frac{d}{2} - 2\gamma > 0 \quad (t \to \infty) \end{aligned} \tag{4.36}$$

The last step in (4.36) uses the Abelian theorem for Laplace transforms (ref. 14, Theorem VIII.2.1) and the fact that $p_r(0,0)=O(r^{-d/2})$ as $r\to\infty$. Finally, from (4.36), $|C^\gamma_{t^{-1}}|=O(t^{-\gamma})$ and the fact that $|e^{\lambda t}|\leqslant e$ for $\lambda\in C^\gamma_{t^{-1}}$, we conclude

$$t^{d/4-p}\int\mu(dw)\left|\int_{C^\gamma_{t^{-1}}}d\lambda\, e^{\lambda t}\left(-\frac{d}{d\lambda}\right)^p\int_0^\infty ds\, e^{-\lambda Ms}\frac{1}{2}\lambda^2\tilde{\mathbf{E}}U_s^2\right|$$
$$=O(t^{-d/4+3-3\gamma})\qquad(t\to\infty)\tag{4.37}$$

So if $\gamma>1-d/12$, then (4.33) follows in $L^1(\mu)$. ▮

### 4.5. Estimates for $C_{t^{-1}}\backslash C^\gamma_{t^{-1}}$

In this section we prove (4.9) in Lemma 6 for the integral restricted to $\lambda\in C_{t^{-1}}\backslash C^\gamma_{t^{-1}}$.

**Lemma 12.** For $p\geqslant 0$ and $\gamma<1-d/4p$

$$\lim_{t\to\infty}t^{d/4-p}\left|\int_{\lambda\in C_{t^{-1}}\backslash C^\gamma_{t^{-1}}}d\lambda\, e^{\lambda t}\left(-\frac{d}{d\lambda}\right)^p[H(\lambda,w)-G(\lambda,w)]\right|=0$$
$$\text{in }\mu\text{-probability}\tag{4.38}$$

*Proof.* We shall show that

$$\left\|\left(-\frac{d}{d\lambda}\right)^p H(\lambda,w)\right\|_{L^2(\mu_0)}+\left\|\left(-\frac{d}{d\lambda}\right)^p G(\lambda,w)\right\|_{L^2(\mu)}\leqslant C\,|\mathrm{Im}\,\lambda|^{-p-1}\qquad\text{for all}\quad p\tag{4.39}$$

Since $|e^{\lambda t}|=e$ for $\lambda\in C_{t^{-1}}$ and since $\mu$ is absolutely continuous w.r.t. $\mu_0$, with bounded Radon–Nikodym derivative, (4.39) implies

$$\left|\int_{\lambda\in C_{t^{-1}}\backslash C^\gamma_{t^{-1}}}d\lambda\, e^{\lambda t}\left(-\frac{d}{d\lambda}\right)^p[H(\lambda,w)-G(\lambda,w)]\right|$$
$$=O_\mu\left(\int_{t^{-\gamma}}^\infty dy\, y^{-p-1}\right)=O_\mu(t^{\gamma p})\tag{4.40}$$

which implies the claim when $d/4-p-\gamma p<0$.

To prove (4.39), we first note that by (1.7)–(1.10) and the first line of (1.23)

$$\begin{aligned}H(\lambda,w)&=(\lambda-L)^{-1}\varphi(w),\qquad &\varphi(w)&=w^{-1}(0)-M^{-1}\\ G(\lambda,w)&=(\lambda M-\tilde{L})^{-1}\kappa(w),\qquad &\kappa(w)&=-\frac{1}{M^2}[w(0)-M]\end{aligned}\tag{4.41}$$

where we also recall (4.1), (4.3), and (4.7). This gives, since $-L$ is self-adjoint and nonnegative on $L^2(\mu_0)$,

$$\begin{aligned}
&\left\| \frac{1}{p!}\left(-\frac{d}{d\lambda}\right)^p H(\lambda, w)\right\|^2_{L^2(\mu_0)} \\
&\quad = \langle (\lambda - L)^{-p-1}\varphi, (\lambda - L)^{-p-1}\varphi\rangle \\
&\quad = \langle \varphi, [(\lambda^* - L)(\lambda - L)]^{-p-1}\varphi\rangle \\
&\quad = \int_0^\infty [(\lambda^* + \gamma)(\lambda + \gamma)]^{-p-1}\, d\alpha_\varphi(\gamma)
\end{aligned} \tag{4.42}$$

where the last equality follows from the spectral theorem, i.e., $d\alpha_\varphi(\gamma)$ is a positive measure with support in $[0, \infty)$ depending on $\varphi$. The importance of the last representation is that we can substitute $(\lambda^* + \gamma)(\lambda + \gamma) \geqslant |\mathrm{Im}\,\lambda|^2$ to obtain

$$\begin{aligned}
&\left\| \frac{1}{p!}\left(-\frac{d}{d\lambda}\right)^p H(\lambda, w)\right\|^2_{L^2(\mu_0)} \\
&\quad \leqslant |\mathrm{Im}\,\lambda|^{-2(p+1)} \int_0^\infty d\alpha_\varphi(s) = |\mathrm{Im}\,\lambda|^{-2(p+1)}\, \|\varphi\|^2_{L^2(\mu_0)}
\end{aligned} \tag{4.43}$$

Finally use that

$$\|\varphi\|^2_{L^2(\mu_0)} = M^{-1}\mathbf{E}_\mu[w^{-1}(0) - M^{-1}] < \infty$$

by (2.1)(ii). The same argument works for the second term in the l.h.s. of (4.39). Use that $\|\kappa\|^2_{L^2(\mu)} = V^2/M^4 < \infty$ by (2.1)(iii). Hence (4.39) follows. ∎

Lemmas 10–12 prove Lemma 6, and thereby Proposition 6.

### 4.6. Proof of Proposition 7

In this section we prove the functional central limit theorem for $\xi_t(w)$ announced in (4.5)–(4.6). We first formulate a lemma showing that in the limit as $t \to \infty$ we may replace $p_t(0, x)$ in (4.3) by its Gaussian limit

$$f_t(x) = \left(\frac{d}{2\pi t}\right)^{d/2} e^{-x^2/2t} \tag{4.44}$$

**Lemma 13.** In $d \geqslant 1$

$$\lim_{t\to\infty} t^{d/4} \sum_x [p_{st}(0, x) - f_{st}(x)][w(x) - M] = 0 \qquad \text{in} \quad L^2(\mu) \tag{4.45}$$

uniformly on compact $t$-intervals.

*Proof.* Compute, using the i.i.d. property of $\mu$,

$$\mathbf{E}_\mu \left\{ t^{d/4} \sum_x [p_{st}(0, x) - f_{st}(x)][w(x) - M] \right\}^2$$

$$= V^2 t^{d/2} \sum_x [p_{st}(0, x) - f_{st}(x)]^2$$

$$\leqslant [2 + o(1)] V^2 \sup_x |t^{d/2} p_{st}(0, x) - f_s(x)| \qquad (t \to \infty) \qquad (4.46)$$

The r.h.s. tends to zero as $t \to \infty$ by the local limit theorem for SRW (ref. 10, p. 77). ∎

Lemma 13 shows that instead of $t^{d/4}\xi_{st}$ it suffices to consider [recall (4.3)]

$$t^{d/4}\xi^*_{st} = -\frac{1}{M^2} \sum_x (t^{-1/2})^{d/2} f_{s/M}(t^{-1/2}x)[w(x) - M] \qquad (4.47)$$

where we substitute $f_{st/M}(x) = t^{-d/2} f_{s/M}(t^{-1/2}x)$.

We shall now put the problem of the weak convergence of the latter expression in the context of a central limit theorem for Schwarz distributions. Indeed, for $\varepsilon > 0$ let $X^\varepsilon_w$ be the random Schwarz functional defined by

$$X^\varepsilon_w(\Phi) = \sum_x \varepsilon^{d/2} \Phi(\varepsilon x)[w(x) - M], \qquad \Phi \in S(\mathbf{R}^d) \qquad (4.48)$$

The distribution of the random variable $X^\varepsilon_w$ is a probability measure $P_\varepsilon$ on the dual Schwarz space $S^*(\mathbf{R}^d)$. In this context the central limit theorem reads (ref. 11, Section 3.1)

$$P_\varepsilon \to \mu_G \qquad \text{weakly as} \quad \varepsilon \to 0 \qquad (4.49)$$

where $\mu_G$ is the Gaussian measure on $S^*(\mathbf{R}^d)$ defined by its characteristic function[8]

$$\int_{S^*(\mathbf{R}^d)} \mu_G(d\Phi^*) \exp(i\langle \Phi^*, \Phi \rangle) = \exp\left[ -\tfrac{1}{2} V^2 \int_{\mathbf{R}^d} \Phi(x)^2 \, dx \right] \qquad (4.50)$$

with $\langle \Phi^*, \Phi \rangle = \Phi^*(\Phi)$ the canonical pairing.

If we consider any one-parameter family of Schwarz functions $\{\Phi_s : s \geqslant 0\}$, then under $\mu_G$ the stochastic process $\{\langle \Phi^*, \Phi_s \rangle : s \geqslant 0\}$ is an $\mathbf{R}$-valued Gaussian process with covariance function given by[8]

$$\int_{S^*(\mathbf{R}^d)} \mu_G(d\Phi^*) \langle \Phi^*, \Phi_r \rangle \langle \Phi^*, \Phi_s \rangle = V^2 \int_{\mathbf{R}^d} \Phi_r(x) \Phi_s(x) \, dx \qquad (4.51)$$

Now return to (4.47) and apply the above formalism with

$$\Phi_s = -\frac{1}{M^2} f_{s/M} \qquad (s > 0)$$
$$\varepsilon = t^{-1/2} \tag{4.52}$$

Then we can conclude that

$$t^{d/4} \xi_{st}^* \to Z_s \qquad \text{weakly as} \quad t \to \infty \tag{4.52}$$

where $\{Z_s : s > 0\}$ is Gaussian with $\mathbf{E} Z_s = 0$ and

$$\begin{aligned}\mathbf{E}[Z_r Z_s] &= \frac{V^2}{M^4} \int_{\mathbf{R}^d} f_{r/M}(x)\, f_{s/M}(x)\, dx \\ &= \frac{V^2}{M^4} f_{(r+s)/M}(0) \\ &= \frac{V^2}{M^4} \left(\frac{dM}{2\pi(r+s)}\right)^{d/2}\end{aligned} \tag{4.54}$$

## ACKNOWLEDGMENTS

We thank P. Scheunders for drawing our attention to ref. 5. This has motivated Theorem 1. We thank H. Spohn for raising the question of the effect of fixing the environment. This has motivated Theorem 2.

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